International Journal of Theoretical Physics, Vol. 40, No. 2, 2001

Duals for Non-Abelian Lattice Gauge Theories by Categorical Methods

Harald Grosse[1] and Karl-Georg Schlesinger[2]

Received March 24, 1999

We introduce duals for non-Abelian lattice gauge theories in dimension at least three by using a categorical approach to the notion of duality in lattice theories. We first discuss the general concepts for the case of a dual-triangular lattice (i.e., the dual lattice is triangular) and find that the commutative tetrahedron condition of category theory can directly be used to define a gauge-invariant action for the dual theory. We then consider the cubic lattice (where the dual is cubic again). The case of the gauge group SU(2) is discussed in detail. We will find that in this case gauge connections of the dual theory correspond to SU(2) spin networks, suggesting that the dual is a discrete version of a quantum field theory of quantum simplicial complexes (i.e. the dual theory lives already on a quantized level in its classical form). We conclude by showing that our notion of duality leads to a hierarchy of extended lattice gauge theories closely resembling the one of extended topological quantum field theories. The appearance of this hierarchy can be understood by the quantum von Neumann hierarchy introduced by one of the authors in previous work.

1. INTRODUCTION

For a lattice gauge theory with Abelian gauge group *G*, the dual is defined as the theory given by the dual lattice and the group of characters of *G* as the gauge group (We will use the term lattice to signify a whole CW-complex of maximal dimension embedded in the manifold we work in, e.g., a triangulation of the manifold.) Here one assumes that starting with an action defined by minimal coupling in the original theory (which we will always do in the sequel), the action of the dual theory is again defined by minimal coupling. Since the group of characters of *G* is again Abelian, the

[1] Institute for Theoretical Physics, University of Vienna, A-1090 Vienna, Austria; e-mail: grosse@doppler.thp.univie.ac.at
[2] Erwin Schrödinger Institute for Mathematical Physics, A-1090 Vienna, Austria; e-mail: schles-@math.uni-wuppertal.de

0020-7748/01/0200-0459$19.50/0

dual theory is of the same type as the old one. It is natural therefore to ask for a notion of dual of a non-Abelian lattice gauge theory, a problem which has been around in the field for at least two decades (We mention the approach of ref. 1 here, but this differs from ours in the fact that it does not start from the observation of the Doplicher–Roberts theorem and therefore does not lead to a hierarchy of extended lattice gauge theories, but leads back to the original theory upon double dualization. In ref. 12, we find the following remark on duality: "the *dual group*, i.e., the set of irreducible representations of G. The dual G^* of an abelian group G has also the structure of an abelian group . . . Such a group structure does not exist in the nonabelian case and this prevents a straightforward generalization of duality." The categorical approach via the Doplicher–Roberts theorem allows for exactly this straightforward generalization.) The first step toward such a notion is simple: If G is a compact group (which we will always assume), for G non-Abelian, the information on G, though not reducible to its characters, is still fully contained in $Rep(G)$, the category of continuous unitary finite-dimensional representations of G (i.e., the objects are representations of this type and the morphisms the intertwiners between these). This is shown by the Doplicher-Roberts theorem [10] and extended to the case of a compact supergroupoid in ref. 4. Using an equivalent categorical formulation of lattice gauge theories given in ref. 5 and observing that $Rep(G)$ is a 2-category with one object, we are led to a dual of a non-Abelian lattice gauge theory as an extended lattice gauge theory (in a sense to be specified below). We give a short sketch of the content of the paper. We always assume G to be a compact Lie group.

In Section 2, we introduce the dual of a non-Abelian lattice gauge theory on a lattice in dimension at least three which is dual to a triangular one (we will call such lattices dual-triangular for short). We introduce an action based on the commutative tetrahedron condition of ref. 17 and prove gauge invariance. Besides this, we give a formal definition of a partition function and discuss the relation to the usual notion of duality in the Abelian case. We show how our approach extends to the case of a cubic lattice.

In Section 3, we discuss the case of the gauge group SU(2) in detail. We show how one can strictly define the partition function by using a normalized sum on $Rep(G)$. We find that gauge connections of the dual theory correspond to SU(2) spin networks which are four-valent in the case of a cubic lattice. So, there is a correspondence of the classical dual theory to quantum simplicial complexes [5], i.e., the dual theory lives already on a quantized level. This fits in with the fact that the action and partition function are purely combinatorial for the dual theory. We argue that the physical meaning of the dual theory is to be seen as some kind of lattice approximation of a quantum field theory of quantum simplicial complexes.

We make a conjecture about the continuum limit of the dual theories in Section 4. Section 5 contains a short discussion of the hierarchy of extended lattice gauge theories which emerges because the dual of the dual (which can be introduced for dimension at least four) is not the theory we started from, but a theory on a still higher level of the hierarchy. This hierarchy and its dependence on dimension closely parallel the hierarchy of extended topological quantum field theories (TQFTs) (see, e.g., ref. 8 and the literature cited therein). We show how this hierarchy can be understood in the framework of quantum set theory [15].

Section 6 contains some concluding remarks and an outlook on possible further work.

2. DEFINITION OF THE DUAL THEORY AND GAUGE INVARIANCE OF THE ACTION

Let a lattice gauge theory with possibly non-Abelian gauge group G in dimension at least three be given, i.e., we have a (finite) lattice γ (given by a set E of edges and a set V of vertices) embedded in a manifold M with dimension $n \geq 3$. A connection is a function $V \to G$ and a gauge transformation is a function $E \to G$. For simplicity, we will assume that there is always at most one edge between a given pair of vertices. So, we can unambigously label vertices by i, j, k, etc., and edges by pairs (i, j), (j, k), etc. A gauge transformation $(h_i)_i$ acts on a connection $(g_{ij})_{ij}$ by

$$g_{ij} \mapsto h_j g_{ij} h_i^{-1}$$

Since we assume minimal coupling, the action S is defined by taking the products of the g_{ij} around minimal 2-cells, e.g., for a triangular lattice, we define $S_\triangle$ as the (normalized) trace of the product of the g_{ij} around $\triangle$ for every triangle $\triangle$ (where reversing the orientation of an edge means replacing g_{ij} by g_{ij}^{-1} and we implicitly assume a fixed representation of G to be given). The action is then defined as

$$S = \sum_\triangle S_\triangle$$

and the partition function Z for every temperature $T = 1/\beta$ by

$$Z = \int d\mu \; e^{-\beta S}$$

where the integral is taken over all connections and the measure is the one induced by the Haar measure on G.

There is an equivalent definition of a lattice gauge theory using the language of category theory [5]. Recall that a category consists of a set of vertices, a set of edges, and a partially defined product of edges (defined whenever the endpoint of the first is the point where the second starts) which is closed under this product and where the vertices are supposed to act as left and right identities under the product (the vertices are also called objects and the edges are called morphisms). Denote by γ again the free category on the lattice (i.e., adjoining all formal products of composable edges). Interpreting G as a category with one object (taking the elements of G as the edges and the product as the product in G), the following definition of connection is equivalent to the usual one: A connection is a functor (i.e., a morphism between categories) from γ to G. To introduce gauge transformations, we need the notion of a natural transformation between two functors $\mathcal{F}$, $\mathcal{G}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$. A natural transformation η is a family $(\eta_c)_c$ of morphisms in $\mathcal{D}$ indexed by objects in $\mathcal{C}$ and satisfying

$$\eta_c\colon\quad \mathcal{F}(c) \to \mathcal{G}(c)$$

and commutativity of

$$\begin{array}{ccc} \mathcal{F}(c) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(d) \\ \eta_c\downarrow & & \downarrow\eta_d \\ \mathcal{G}(c) & \xrightarrow[\mathcal{G}(f)]{} & \mathcal{G}(d) \end{array}$$

for every morphism $f\colon c \to d$ in $\mathcal{C}$ (where we use as usual the same functor symbol for the images of objects and morphisms). The commutativity of the diagram expresses the fact that the natural transformation shifts the image of $\mathcal{C}$ under $\mathcal{F}$ in the one under $\mathcal{G}$ compatible with the categorical structure. A natural equivalence is an invertible natural transformation, i.e., one where all the η_c are invertible. A gauge transformation is then a natural equivalence between two functors representing connections.

We will now assume that γ is a dual-triangular lattice, i.e., the dual lattice (in the sense of Poincaré duality) is triangular. The tensor product of representations gives a monoidal structure on *Rep*(G) (a monoidal category is one carrying a functorially given product which satisfies the axioms of a monoid up to isomorphisms; see any introductory text on category theory for the details). This means *Rep*(G) is a monoid in the category **Cat** of (small) categories and functors. Since a usual monoid—and therefore also a group—is a monoid in the category **Set** of sets and functions, this suggests that we should shift all the structural elements of a lattice gauge theory from the level of sets to the level of categories in order to define the dual of a non-Abelian lattice gauge theory as an extended lattice gauge theory (as we will call the generalizations gained this way) with "gauge monoid" *Rep*(G).

Since a usual lattice gauge theory is defined by functors and natural transformations, we have to invoke notions of 2-category theory in order to make this precise.

The prototype of a category is the category **Set** of sets and functions. The prototype of a 2-category is the category **Cat** of small categories and functors. **Cat** has more structure on it then a simple category because we have natural transformations between functors. This can be viewed in the following way: The extra structure implies that every morphism set

$$Hom(C, D)$$

in **Cat** is actually not only a set, but a category itself where composition and identities in **Cat** are compatible with this categorical structure on the *Hom*-sets (i.e., composition and identities are functorial with respect to the structure on the *Hom*-sets). A general category with this kind of extra structure is called a 2-category. The usual morphisms are called 1-morphisms; the morphisms in the *Hom*-sets are called 2-morphisms. A morphism between 2-categories is called a 2-functor. Obviously, there is a notion of natural transformation between 2-functors again. For two parallel 2-functors $\mathcal{F}$ and $\mathcal{G}$ between 2-categories $\mathcal{C}$ and $\mathcal{D}$, this consists of an assignment of 1-morphisms of $\mathcal{D}$ to objects of $\mathcal{C}$ and an assignment of 2-morphisms of $\mathcal{D}$ to 1-morphisms of $\mathcal{C}$ satisfying the commutativity conditions analogous to the above one (which again just express the fact that the images of the 2-functors are shifted in a way compatible with the 2-categorical structure). See ref. 11 for an introduction to so called higher-dimensional category theory or ref. 6.

In the same way in which a group can be understood as a category with one object, a monoidal category like $Rep(G)$ can be understood as a 2-category with one object: The objects of $Rep(G)$ give the 1-morphisms, the product of 1-morphisms (which is universally defined since there is only one object) is defined as the tensor product $\otimes$, and the intertwiners in $Rep(G)$ give the 2-morphisms (with the usual composition $\circ$ as product). We denote this 2-category again by $Rep(G)$. Let σ be the 2-category generated by the dual lattice, i.e., we take vertices $i, j, k, \ldots$ of the dual lattice as objects, edges $ij, jk, \ldots$ as 1-morphisms, and triangles like ijk as 2-morphisms and take the closure under formal compositions. We make the convention that 2-morphisms in triangles are oriented in one of the two ways in Fig. 1 (i.e., the 2-morphism goes either from two edges to the third or vice versa; observe that we do not consider the third as the composite of the other two when defining σ, but we take all formal composites), and that the four 2-morphisms on the faces of a tetrahedron follow the orientation in Fig. 2.

We will see that this orientation is necessary in order to introduce a minimal coupling action.

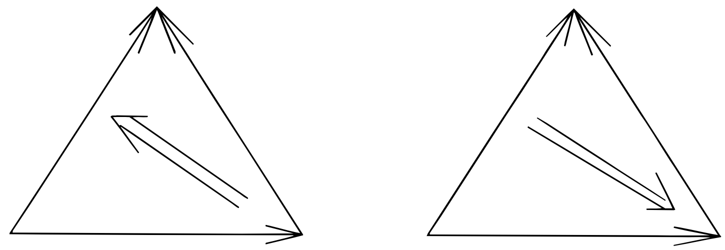

Fig. 1. Orientation of morphisms in triangles.

A connection of the dual theory is then a 2-functor from σ to *Rep*(*G*) and a gauge transformation is a natural transformation between two such functors with all its 2-morphisms invertible [it does not make sense to require invertibility of the 1-morphisms of the natural transformation, too, since *Rep*(*G*) is monoidal with ⊗, but not a group-like structure]. Let us write down in detail what this means: A connection is a coloring of the edges with continuous unitary finite-dimensional representations V_{ij} and the faces (i.e., triangles) with intertwiners A_{ijk} subject to the orientation requirement given above. Besides this, the functoriality of the coloring introduces the requirement that only representations on the three edges of a triangle can appear which allow for an intertwiner. A gauge transformation is a coloring of the vertices with continuous unitary finite-dimensional representations U_i and the edges with invertible intertwiners B_{ij}. A gauge transformation between connections (V_{ij}, A_{ijk}) and (V'_{ij}, A'_{ijk}) has to satisfy the following requirements (following from the definition of a natural transformation):

$$V'_{ij} \otimes U_j \cong U_i \otimes V_{ij}$$

where

$$B_{ij}: \quad U_i \otimes V_{ij} \to V'_{ij} \otimes U_j$$

explicitly gives the isomorphism. Besides this, we have the requirement

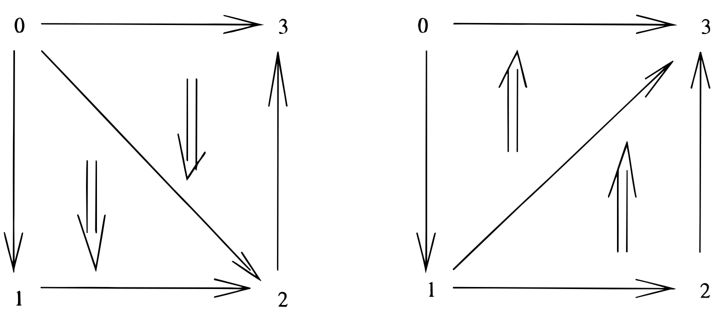

Fig. 2. Orientation of morphisms for the tetrahedron.

$$B_{13} \circ (id_{U_1} \otimes A_{123}) = (A'_{123} \otimes id_{U_3}) \circ \widetilde{B_{13}}$$

for the intertwiners of a triangle with vertices 1, 2, 3, where $\widetilde{B_{13}}$ is defined as the composite of

$$B_{12} \otimes id_{V_{23}}: \quad U_1 \otimes V_{12} \otimes V_{23} \to V'_{12} \otimes U_2 \otimes V_{23}$$

and

$$id_{V'_{12}} \otimes B_{23}: \quad V'_{12} \otimes U_2 \otimes V_{23} \to V'_{12} \otimes V'_{23} \otimes U_3$$

We have written the condition for the intertwiners for a triangle oriented in the first of the two above ways (i.e., with the arrow on the face ingoing to the third); for the second one, one gets obvious orientation reversals of the morphisms.

Having defined gauge connections and gauge transformations, we now proceed to introduce an action. We start from the commutative tetrahedron condition as given in ref. 17, which was already implicit in our orientation requirement for the tetrahedron, above. Commutativity of a colored tetrahedron is shown in Fig. 3.

Requiring the intertwiners on the faces in the right one of the two squares to be in reversed orientation (as we did above), we get the possibility to compose the intertwiners around the tetrahedron and then to take the trace. This leads to the following expression for each tetrahedron T:

$$S_T = \mathrm{tr}(A_{013} \circ (id_{V_{01}} \otimes A_{123}) \circ (id_{V_{23}} \otimes A_{012}) \circ A_{023})$$

(where again 0, 1, 2, 3 are the vertices of the tetrahedron).

Lemma 1. S_T is gauge invariant.

Proof. We have to consider a connection (V'_{ij}, A'_{ijk}) which is a gauge transform of (V_{ij}, A_{ijk}) under a gauge transformation (U_i, B_{ij}). We have

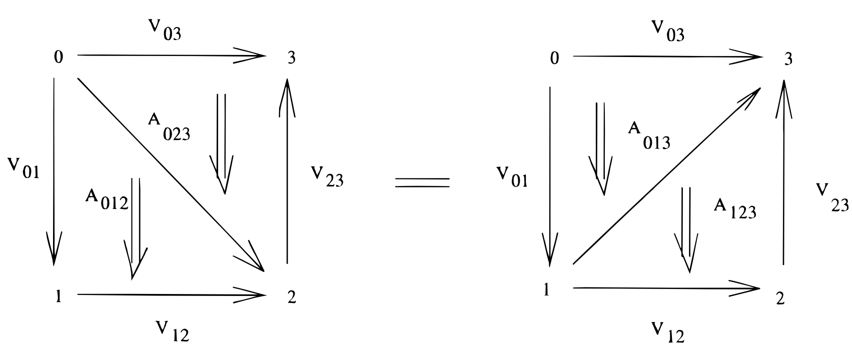

Fig. 3. Commutativity of a colored tetrahedron.

$$S'_T = \mathrm{tr}(A'_{013} \circ (id_{V'_{01}} \otimes A'_{123}) \circ (id_{V'_{23}} \otimes A'_{012}) \circ A'_{023})$$

and since we use normalized traces (i.e., the trace of the identity is one), we can take the tensor product with the identity on U_3, getting

$$S'_T = \mathrm{tr}\begin{pmatrix} (A'_{013} \otimes id_{U_3}) \circ (id_{V'_{01}} \otimes A'_{123} \otimes id_{U_3}) \\ \circ\, (id_{V'_{23}} \otimes A'_{012} \otimes id_{U_3}) \circ (A'_{023} \otimes id_{U_3}) \end{pmatrix}$$

which gives

$$S'_T = \mathrm{tr}\begin{pmatrix} \widetilde{B_{03}^{-1}} \circ (A'_{013} \otimes id_{U_3}) \circ (id_{V'_{01}} \otimes A'_{123} \otimes id_{U_3}) \\ \circ\, (id_{V'_{23}} \otimes A'_{012} \otimes id_{U_3}) \circ (A'_{023} \otimes id_{U_3}) \circ \widetilde{B_{03}} \end{pmatrix}$$

By calculation, one proves

$$S'_T = S_T$$

which concludes the proof. ■

Remark 1. We make a short remark concerning the above lemma: We implicitly used the symmetry of the tensor product on $Rep(G)$. If one would work with a category of representations of a quantum group, one would have to check that one can insert the twists in the right places (which probably involves inserting twists in the definition of the action).

We now define the action as

$$S = \sum_T S_T$$

where the sum is to be taken over all tetrahedra in the dual lattice. For β the inverse of absolute temperature, we introduce the partition function Z as

$$Z = \sum_{\text{representations}} \mu \sum_{A_{ijk}} e^{-\beta S}$$

Here $\Sigma_{\text{representations}}\ \mu$ means the sum over all possible colorings of edges by irreducible continuous unitary finite-dimensional representations of G subject to the condition that there exist always intertwiners on the triangles. For simplicity, we restrict colorings to irreducible representations in the sequel, but how to proceed in the general case is obvious then. The expression μ indicates that we take a weighted sum in general, in order to get convergence (which means a kind of normalization). The second sum runs over all intertwiners which can be inserted on the triangles for fixed representations. For a compact Lie group and a finite lattice it is always finite.

Remark 2. Observe that the double sum in the definition of Z is the analog of the Haar measure in conventional lattice gauge theories, i.e., our dual theories are purely combinatorial, which should make them especially amenable to computational approaches.

Remark 3. For the special case of G Abelian, all the information of $Rep(G)$ is, of course, contained in the characters. But if one decides to use, in spite of this, the full category of representations, following the approach outlined here, one still gets a different dual because the dual in our case is an extended lattice gauge theory where one has the additional requirement that faces are colored, too. But one can argue that proceeding this way is not natural because in the Abelian case, $Rep(G)$ is only artificially considered as a category. So, our construction should be restricted to truely non-Abelian groups (see also our discussion in the context of the quantum von Neumann hierarchy below, where this can be understood in more rigorous terms).

Remark 4. It is clear how to get observables, i.e., more general gauge-invariant expressions then the partition function: The partition function is gained by combining intertwiners on faces around tetrahedra. More generally, one can combine intertwiners on faces along larger closed bodies in the lattice (by pasting tetrahedra together).

In concluding this section, we briefly discuss the case of a cubic lattice γ where the dual is cubic again. Denote by 1–8 the vertices of a cube. Orient the edges of a square alternating as ingoing and outgoing, as indicated in Fig. 4 for one of the squares of the cube, and attach an intertwiner from the tensor product of the representations on the ingoing edges to the tensor product on those of the outgoing edges to the square. Choose the orientations on all the squares of the cube in such a way that they are compatible with composing around the cube to obtain the following expression for each cube K:

$$A_K = (id_{V_{14}} \otimes A_{5678} \otimes id_{V_{23}}) \circ (A_{1478} \otimes A_{2356})$$
$$\circ (A_{1257} \otimes A_{3468}) \circ (id_{V_{57}} \otimes A_{1234} \otimes id_{V_{68}})$$

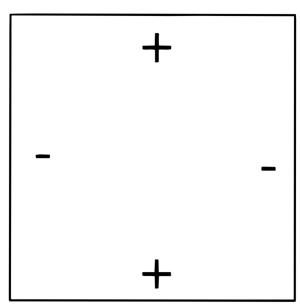

Fig. 4. Orientation of the square for the cubic lattice.

where we have suppressed the appearance of twists interchanging factors in the tensor product (since it is symmetric). Now, define

$$S_K = \operatorname{tr} K$$

and define the action and partition function as above, using S_K instead of S_T.

3. THE CASE OF SU(2)

We are now going to consider the case $G = \mathrm{SU}(2)$ in detail. We assume again that γ is dual-triangular. Since the representations of SU(2) are labeled by spin numbers j, assuming that N is the number of edges of the dual lattice, we can write the partition function as

$$Z = \sum_{j_1,\ldots,j_N} \mu \sum_{A_{ijk}} e^{-\beta S}$$

where the first sum is again constrained by the existence of intertwiners on the faces. By Clebsch–Gordan theory, it follows that for a triangular dual lattice the intertwiners are fixed, once the representations are. So, the second sum is trivial in this case. Choosing

$$\mu = \frac{1}{(j_1 \cdot \cdots \cdot j_N)^2}$$

we obviously get convergence. With this choice, we then have the partition function

$$Z = \sum_{j_1,\ldots,j_N} \frac{1}{(j_1 \cdot \cdots \cdot j_N)^2}\, e^{-\beta S}$$

Let us now proceed to discuss the physical meaning of the dual theory. To achieve this, we first remark that the connections of the dual theory can equivalently be interpreted as SU(2) spin networks. This is most easily seen in the following way: Replace each triangle by a point and each edge by a new edge intersecting it (which means considering the CW-complex consisting of the vertices, edges, and faces of the dual lattice and taking again the Poincaré dual of this) and take the coloring induced by the one of the dual lattices, i.e., we get representations on the edges and intertwiners on the vertices connecting these. But this is precisely the definition of a spin network (see e.g., refs. 7 and 14 for the origin of the concept). In this case, the spin networks arising are trivalent, i.e., we have three edges at every vertex. Using a cubic lattice, we arrive at four-valent spin networks, which are discussed in detail in ref. 7, because every vertex of a four-valent spin network can be seen as containing the information on a quantum tetrahedron (the spins on every vertex describing states of the bivectors on the four faces of a tetrahedron and the intertwiner on the vertex giving the condition that the tetrahedron is closed). So, a whole spin network gives a state of a quantum complex (and the type of a spin network, i.e., trivalent, four-valent, etc., gives the type of complex; for a four-valent spin network, we have a state of a quantum simplicial complex). See, e.g., ref. 7 for the details of this kind of quantum geometry and ref. 9 for a detailed account of the quantum tetrahedron.

In conclusion, we have that each gauge connection of the dual theory corresponds to a spin network and therefore to a state of a quantum complex. This means that the dual theory lives already in the classical case on a quantized level, and naturally fits in with the fact that the action takes a purely combinatorial form. Taking the sum over connections in the partition function then means taking the sum over quantum states. This is the reason why we suspect that the natural interpretation of the dual theory is some kind of lattice version of a quantum field theory of quantum geometry. We will deal with this point again in the next section.

4. A CONJECTURE ABOUT THE CONTINUUM LIMIT

We restrict again to $G = \mathrm{SU}(2)$ in this section. The continuum limit of a conventional lattice gauge theory is believed to be given by a gauge theory with connections living on a principal bundle over the manifold M with structure group G. Now, our "structure group" is $Rep(G)$, i.e., a continuum version of the dual theory consists of a kind of bundle over M with fibers given by $Rep(G)$. The monoidal structure on $Rep(G)$ gives an immediate definition of transition functions by left multiplication.

Remark 5. The usual compatibility condition for the transition functions (which is in principle a 1-cocycle condition) should in a categorical setting only be required in a weak form, i.e., up to a natural equivalence. Then a so-called coherence condition (here a commutative tetrahedron for the natural equivalences) is needed in order to assure that the weak condition can be used in a similar way as the strict one. Since we did not deal with the weak forms of categorical conditions in the whole paper, but assumed implicitly that we work in a strictif setting (roughly speaking, having divided out all the isomorphisms, which is possible in a 2-categorical setting, but not in general in still higher dimensional categories), we will not deal with this question here and refer the reader to ref. 16, where we already considered bundles with categorical fibers.

Now, a connection in usual gauge theory is a Lie algebra-valued 1-form, arising in the following way: The analog of the coloring of the edges by group elements gives a function g with values $g(x) \in G$ in the continuum case. Developing this (by assuming that it is differentiable) around points $x \in M$ leads to the connection. This means that in our case, we should expect a pair of functions j and A with values irreducible, continuous, finite-dimensional unitary representations and certain intertwiners, respectively, which arise from the coloring of edges and faces. Since both irreducible representations and intertwiners are parametrized discretely for SU(2), we conclude that even upon requiring only continuity, j and A have to be constant. We therefore arrive at the following conjecture:

Conjecture 2. Requiring continuity, there is no nontrivial continuum limit in the sense of a theory given in terms of connections on a kind of principal bundle for the dual of the minimal coupling SU(2) lattice gauge theory.

There is a second possibility to approach the question of a continuum limit which inherently dispenses with the continuity requirement, namely the projective limit approach of Ashtekar and Lewandowski [2, 3]. There one takes a projective limit over all SU(2) lattice gauge theories on all possible lattices (with analytic edges; analyticity is assumed for technical reasons there, but an extension to the smooth case is supposedly possible) and arrives in the continuum at a theory where a gauge connection is an arbitrary function which assigns elements of SU(2) to analytic paths in M and respects the requirements that composition and orientation reversal go into the corresponding group operations. Gauge transformations are simply SU(2)-valued functions on M, operating on gauge connections in the obvious way. We therefore make the conjecture below about a projective limit over all duals of all SU(2) lattice gauge theories in M. But before we can do this, we need one more definition.

Definition 1. We call a sheet a two-dimensional manifold immersed into M together with a boundary which is the union of a finite collection of oriented paths (called edges of the sheet), carrying an additional marker $+$ or $-$ (signifying them as ingoing or outgoing with respect to the sheet). The endpoints and beginning points of the edges (where the boundary may be singular) are called the vertices of the sheet.

Conjecture 3. We suppose that a projective limit over all duals of all SU (2) lattice gauge theories in M will be of the following kind: A gauge connection is a (not necessarily continuous) coloring of edges of sheets by irreducible, continuous, finite-dimensional unitary representations and sheets by intertwiners between the tensor product of the representations on the ingoing edges and the tensor product on those on the outgoing edges, subject to the requirement that pasting of sheets leads to composition of intertwiners and the disjoint union of sheets goes to the tensor product of intertwiners.

A gauge transformation is a (again not necessarily continuous) coloring of vertices by representations and edges by intertwiners, acting on a gauge connection again by the laws of a natural transformation.

Remembering the discussion in the foregoing section, we suspect that a theory of the kind conjectured above can be gained as a second-quantized theory from the Ashtekar–Lewandowski approach.

5. THE HIERARCHY OF (EXTENDED) LATTICE GAUGE THEORIES

The central question in this section is the following: Can we take the dual of the dual of a non-Abelian lattice gauge theory and what does this theory look like? We should remark from the beginning that in this section we do not want to spell out a detailed definition, but sketch the rough form of the overall scheme. Taking the dual of the dual lattice leads, of course, back to the original lattice. So, the main question is, What is the dual of $Rep(G)$? One has to be very careful in keeping concepts apart here, in order to give the correct answer. As a category of representations, the dual of $Rep(G)$ is, of course, the non-Abelian group G. But we considered $Rep(G)$ as a monoid in **Cat** here, i.e., it played itself the role of the gauge group. So, dualizing again in the way we did when going from G to $Rep(G)$, we have to consider representations of $Rep(G)$ (i.e., functors preserving the monoidal structure) on a 2-vector space. A 2-vector space is a finite-dimensional module category (a category carrying functors for addition and scalar multiplication satisfying axioms analogous to the module axioms) over the category **Hilb** of finite-dimensional Hilbert spaces (i.e., the finite-dimensional Hilbert spaces play the role of the scalars). We refer the reader to refs. 8 and 13 for the technical details of this notion. Actually, we should use 2-Hilbert spaces (as introduced in ref 4) instead of 2-vector spaces, in order to be able to formulate a notion of unitary representation.

Now, all these representations of $Rep(G)$ form a 2-category again, which we denote by $Rep^2(G)$. This 2-category $Rep^2(G)$ then has to be used as the "gauge group" of the dual of the dual theory. In order to be able to give sense to this, we have to assume that we are working in dimension at least four. For simplicity, we assume that the dual of the dual (i.e., the original) lattice is now given by a triangulation of M. We have to introduce the notion of a 3-category at this point, which is defined as a category with all the *Hom*-sets carrying the structure of a 2-category such that composition is compatible with this structure on the *Hom*-sets. A 3-category then has an additional structural level of 3-morphisms. Attaching an arrow to each tetrahedron leads to the free 3-category on the lattice. Besides this, $Rep^2(G)$ carries a kind of monoidal structure again, so it is naturally a 3-category with one object. It is then clear how to introduce gauge connections and gauge transformations of the dual of the dual by 3-functors and natural transformations between these. The action is then defined by composing around the 4-simplex (observe that gauge invariance of the action of the dual above merely depended on the fact that a tetrahedron has a closed surface, which is true for the 4-simplex, too).

If the dimension of M is sufficiently large, we can take the dual many times, each time entering a higher level of categorification. A whole hierarchy of extended lattice gauge theories emerges this way, which closely resembles the hierarchy of extended TQFTs (see, e.g., ref. 8 for an introduction). The hierarchy of extended TQFTs emerges on including singularities of higher and higher codimension in the definition of the cobordism category and here, too, the dimension of the manifolds considered determines the maximal level of categorification. The same is true for the categorified manifolds (gained as non-Abelian cohomology classes over classical manifolds) introduced in ref. 16. In this case, a connection to bundles with categorical fibers and lattice gauge theories was already pointed out.

There is a possibility to understand the emergence of these hierarchies in a unified way. In ref. 15 it was shown how to take the first steps toward mathematics in a set theory where the internal logic is not the classical, one but is given by a q-bit (the lattice of linear subspaces of a two dimensional Hilbert space). It was found there that the analog of the classical von Neumann hierarchy (where sets of higher and higher level emerge from the empty set by successive application of the power set operation)—which we called the quantum von Neumann hierarchy—is given by the hierarchy of n-categories. This means the hierarchies of extended lattice gauge theories, extended TQFTs, etc., emerge on climbing to higher and higher levels of the quantum von Neumann hierarchy. The conventional lattice gauge theories are then just the ones we see when remaining on the lowest level of quantum set theory. We now also can understand from an abstract view point why extended lattice gauge theories appear to live on a quantized level already in their classical form: Categorification corresponds to quantization—understood in a very abstract sense—and a kind of iterative quantization is just the meaning of the von Neumann hierarchy in a quantum set theory.

6. CONCLUSION

We have seen that the fact that $Rep(G)$ is a monoid in the category **Cat** of (small) categories and functors naturally suggests a definition of the dual of a non-Abelian lattice gauge theory. Since we have to invoke the language of 2-categories for this, one gets a constraint on the dimension (it has to be at least three in order to have volumes available around which to take the action). For the case $G = \mathrm{SU}(2)$, we found that classical gauge connections of the dual theory already correspond to spin networks, i.e., to states of quantum geometries. This suggests that the dual theory is a lattice version of a quantum field theory of geometry. We made a conjecture about the continuum limit and discussed how the fact that the dual of the dual is not the theory we started from leads to a hierarchy of extended lattice gauge theories. Let us now make a few remarks about some questions deserving further detailed study.

- There is, of course, the question of matter coupling. In conventional lattice gauge theories, this is achieved the following way: As we already remarked at the beginning, one implicitly assumes that a choice of representation of G has been made. While for the pure gauge theory case, this is not relevant, for matter couplings, it becomes decisive. The matter coupling is mathematically represented as an additional coloring of the vertices with vectors of the space where the representation lives. A matter field $(\Psi_i)_i$ couples to a gauge connection $(g_{ij})_{i,j}$ as

$$\langle \Psi_j | g_{ij} | \Psi_i \rangle$$

where $\langle\rangle$ denotes the inner product on the representation space. For a matter coupling in the dual theory, one therefore first has to choose a representation of $Rep(G)$ on a 2-Hilbert space. One then has to use an additional coloring of the vertices, respectively edges, with objects and morphisms of the 2-Hilbert space, and the matter fields couple to the gauge connection by the inner product (which is a functor now, i.e., allows for a simultaneous coupling of objects and morphisms). The general structure of such a coupling can easily be seen: Since a 2-Hilbert space has tuples of finite-dimensional Hilbert spaces (which can, up to isomorphism, be seen as tuples of natural numbers) as objects and tuples of linear operators between these as morphisms, we have a coloring with tuples of spins at the vertices and tuples of operators at the edges.
- Since we work in a 2-categorical setting, there is an additional level of structure which we did not use, namely so-called modifications which are transformations between natural transformations (see any of the introductory texts on higher dimensional category theory mentioned above). This means we can link gauge transformations together by gauge transformations of a second level (gauge transformations on gauge transformations). The physical meaning of this has still to be explored.
- The monoidal structure on $Rep(G)$ is symmetric, i.e., the "gauge group" of the dual theory is Abelian even for non-Abelian G. A non-Abelian dual theory can only be gained when replacing G with a Hopf-algebra. This

means that on climbing up the ladder of extended lattice gauge theories, a *q*-deformed theory is transformed into one with a usual monoid-like (but non-Abelian) gauge object, while a non-Abelian theory is transformed into an Abelian one. Again, this appears as natural when one remembers that the dual theory lives on a higher quantized level (higher level in the quantum von Neumann hierarchy).

ACKNOWLEDGMENTS

We thank H. Gausterer for a discussion and F. Faber for suggestions on the literature and help with the computer system. K.G.S. acknowledges support by research grant D/97/19044 under the Gemeinsames Hochschulsonderprogramm III von Bund und Ländern from the DAAD (German Academic Exchange Service) and the hospitality of the Erwin Schrödinger Institute for Mathematical Physics and the Institute of Theoretical Physics of the University of Vienna during a research stay. The work of H.G. was part of Project P11783-PHY of the Fonds zur Förderung der wissenschaftlischen Forschung in Österreich.

REFERENCES

1. R. Anishetty, S. Cheluvaraja, and H. S. Sharatchandra (1992). Dual of 3-dimensional pure *SU*(2) lattice gauge theory and the Ponzano–Regge model, hep-lat@ftp/9210024.
2. A. Ashtekar and J. Lewandowski (1994). Differential geometry on the space of connections via graphs and projective limits, preprint, CGPG-94/12-4.
3. A. Ashtekar and J. Lewandowski (1996). Quantum theory of geometry I: Area operators, preprint, gr-qc/9602046v2.
4. J. C. Baez (1996). Higher-dimensional algebra II: 2-Hilbert spaces, preprint, q-alg/9609018v2.
5. J. C. Baez (1996). *Adv. Math.* **117**, 253.
6. J. C. Baez (1997). An introduction to *n*-categories, preprint, q-alg/9705009.
7. J. C. Baez (1997). Spin foam models, preprint.
8. J. C. Baez and J. Dolan (1995). *J. Math. Phys.* **36**, 6073.
9. A. Barbieri (1997). Quantum tetrahedra and simplicial spin networks, preprint, gr-qc/9707010.
10. S. Doplicher and J. Roberts (1989). *Invent. Math.* **98**, (1989) 157.
11. J. W. Gray (1974). *Formal Category Theory*, (Springer, Berlin).
12. C. Itzykson and J.-M. Drouffe (1989). *Statistical Field Theory*, Vol. 1 (Cambridge University Press, Cambridge).
13. M. M. Kapranov and V. A. Voevodsky (1994). In *Algebraic Groups and Their Generalizations*, W. J. Haboush and B. J. Parshall, eds. (American Mathematical Society, Providence, Rhode Island), pp. 177–259.
14. R. Penrose (1971). In T. Bastin, *Quantum theory and beyond* (Cambridge University Press, Cambridge), pp. 151–180.

15. K.-G. Schlesinger (n.d.). Towards quantum mathematics, Part I: From quantum set theory to universal quantum mechanics, preprint, Erwin Schrödinger Institute for Mathematical Physics, Vienna, 537 (available at http://www.esi.ac.at).
16. K.-G. Schlesinger (n.d.). Towards quantum mathematics, Part II, preprint, Erwin Schrödinger Institute for Mathematical Physics, Vienna, 556 (available at http://www.esi.ac.at).
17. R. Street (1987). *J. Pure Appl. Algebra* **49**, 283.